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Boundary conditions for scaled random matrix ensembles in the bulk of the spectrum

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Abstract
A spectral average which generalizes the local spacing distribution of the eigenvalues of random $N \times N$ Hermitian matrices in the bulk of their spectrum as $N \to \infty$ is known to be a τ-function of the fifth Painlevé system. This τ-function, $\tau(s)$, has generic parameters and is transcendental but is characterized by particular boundary conditions about the singular point $s = 0$, which we determine here. When the average reduces to the local spacing distribution we find that the τ-function is of the separatrix, or partially truncated type.

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1. Motivations

Recent studies [13–15] have revealed that a generalized spectral average for the universality classes of scaled random Hermitian matrix ensembles—the bulk, hard edge and soft edge classes for $\beta = 2$—is determined by general transcendental solutions of the Painlevé equations $P_{\rm V}$, $P_{\rm III}$ and $P_{\rm II}$ respectively. These works unified earlier studies [19, 23–25] in the sense that this average includes local eigenvalue spacing distributions and moments of the characteristic polynomials as special cases, and extended them in that the parameter sets of the Painlevé equations were exhausted. As a consequence of this identification the logarithmic derivatives of the spectral averages are Jimbo–Miwa–Okamoto σ-functions satisfying second-order second-degree ordinary differential equations. It was noted in the first mentioned works that while identifying the precise parameters of the Painlevé transcendents involved, the algebraic approach employed could not uniquely specify adequately the boundary conditions that the particular solutions to the differential equations should satisfy. It is the purpose of this paper to rectify this for the bulk scaling case and completely determine the boundary conditions for the differential equations of the σ-form.

The plan of our work is as follows. In section 2 we review the isomonodromic formulation of P_{VI} for generic values of the parameters, give a specific parametrization of its monodromy matrices and the local expansion of the τ-function about the fixed singularity $t = 0$. In section 3 we give the corresponding expansion for a generalized spectral average of the circular unitary ensemble, the *spectrum singularity ensemble* and identify its monodromy data by comparison with the previous section. In section 4 we review the necessary isomonodromic formulation for P_V in a parallel manner to the treatment of P_{VI} in section 2. We take the bulk scaling limit of our spectrum singularity average via a formal argument in section 5 and identify the resulting monodromy data applying in this case. We show that these data are consistent with the rigourous theory of the limit transition from P_{VI} to P_V as developed in Kitaev [20]. We wish to advise the reader that we re-use the same symbols in the context of the monodromy theory of P_{VI} as for that of P_V where there is no risk of confusion in order not to burden the notation unnecessarily. Where both are discussed together then we make a notational distinction.

2. Isomonodromy deformation formulation for P_{VI}

Following the conventions and notations of [18, 20] we consider the Lax pair of linear 2×2 matrix ODEs for $\Psi(\lambda; t)$ with four regular singularities in the λ-plane denoted by $\nu \in \{0, t, 1, \infty\}$:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Psi = \left(\frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1} + \frac{A_t}{\lambda - t}\right)\Psi, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Psi = -\frac{A_t}{\lambda - t}\Psi. \tag{2.1}$$

We adopt the convention that the residue matrices A_ν satisfy

$$A_0 + A_t + A_1 = -A_\infty = -\frac{\theta_\infty}{2}\sigma_3, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \theta_\infty \in \mathbb{C}\backslash\mathbb{Z}, \tag{2.2}$$

and the constraints $\operatorname{tr} A_\nu = 0$, $\det A_\nu = -\frac{1}{4}\theta_\nu^2$, $\nu \in \{0, t, 1\}$, defining the formal exponents of monodromy θ_ν. The *τ-function* for P_{VI} is defined as

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\tau = \operatorname{Tr}\left(\frac{A_0}{t} + \frac{A_1}{t-1}\right)A_t, \tag{2.3}$$

and the *σ-function* as

$$\zeta(t) = t(t-1)\frac{\mathrm{d}}{\mathrm{d}t}\log\tau + \frac{1}{4}\left(\theta_t^2 - \theta_\infty^2\right)t - \frac{1}{8}\left(\theta_t^2 + \theta_0^2 - \theta_\infty^2 - \theta_1^2\right), \tag{2.4}$$

which satisfies the second-order second-degree differential equation

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\zeta\left(t(t-1)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\zeta\right)^2 + \left[2\frac{\mathrm{d}}{\mathrm{d}t}\zeta\left(t\frac{\mathrm{d}}{\mathrm{d}t}\zeta - \zeta\right) - \left(\frac{\mathrm{d}}{\mathrm{d}t}\zeta\right)^2 - \frac{1}{16}\left(\theta_t^2 - \theta_\infty^2\right)\left(\theta_0^2 - \theta_1^2\right)\right]^2 \\
&\quad = \left(\frac{\mathrm{d}}{\mathrm{d}t}\zeta + \frac{1}{4}(\theta_t + \theta_\infty)^2\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\zeta + \frac{1}{4}(\theta_t - \theta_\infty)^2\right) \\
&\qquad \times \left(\frac{\mathrm{d}}{\mathrm{d}t}\zeta + \frac{1}{4}(\theta_0 + \theta_1)^2\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\zeta + \frac{1}{4}(\theta_0 - \theta_1)^2\right).
\end{aligned} \tag{2.5}$$

Furthermore, we suppose that the matrices A_ν, $\nu \in \{0, t, 1\}$ are diagonalizable, i.e. that there exist invertible matrices $R_\nu \in SL(2, \mathbb{C})$ such that $R_\nu^{-1} A_\nu R_\nu = \frac{1}{2}\theta_\nu\sigma_3, \theta_\nu \in \mathbb{C}\backslash\mathbb{Z}$. In the neighbourhood of a regular singularity, $\Psi(\lambda)$ can be expanded as

$$\Psi(\lambda) = \sum_{m=0}^{\infty} \Psi_{m\nu}(\lambda - \nu)^{m + \frac{\theta_\nu}{2}\sigma_3} C_\nu, \tag{2.6}$$

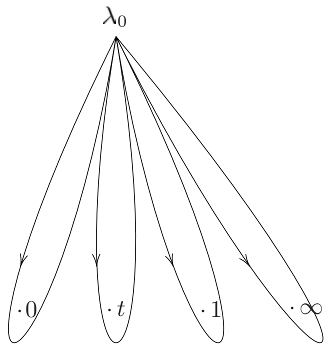

Figure 1. Monodromy representation of the fundamental group for $\mathbb{C}\backslash\{0, t, 1, \infty\}$.

for $\nu \in \{0, t, 1\}$ and for $\lambda = \infty$ in the form

$$\Psi(\lambda) = \left(I + \sum_{m=1}^{\infty} \Psi_{m\infty}\lambda^{-m}\right)\lambda^{-\frac{\theta_\infty}{2}\sigma_3}. \tag{2.7}$$

The monodromy matrices $M_\nu(\nu \in \{0, t, 1, \infty\})$ are defined by $\Psi|_{\nu+\delta\,\mathrm{e}^{2\pi\mathrm{i}}} = \Psi|_{\nu+\delta} M_\nu$, and are given in terms of the connection matrices C_ν by

$$M_\nu = C_\nu^{-1}\,\mathrm{e}^{\pi\mathrm{i}\theta_\nu\sigma_3} C_\nu, \qquad C_\infty = I. \tag{2.8}$$

They satisfy the cyclic relation which in our conventions is taken to be

$$M_\infty M_1 M_t M_0 = I, \tag{2.9}$$

and corresponds to the particular basis of loops given in figure 1

The monodromy data $\mathcal{M}_{\mathrm{VI}} := \{\theta_\nu, C_\nu, M_\nu | \nu = 0, t, 1, \infty\}$ are preserved under deformations with respect to t. The invariants of the monodromy data are defined to be $p_\mu = 2\cos\pi\theta_\mu := \mathrm{Tr}\, M_\mu, \mu \in \{0, t, 1, \infty\}$ and $p_{\mu\nu} = 2\cos\pi\sigma_{\mu\nu} := \mathrm{Tr}\, M_\mu M_\nu, \mu, \nu \in \{0, t, 1\}$.

Now we are in a position to present the details of a parametrization of the monodromy matrices and the expansions of the τ-function about the singular points $0, 1, \infty$ given in Jimbo's study [18]. In this work, Jimbo states the following conditions under which his results apply:

$$\theta_0, \theta_t, \theta_1, \theta_\infty \notin \mathbb{Z}, \tag{2.10}$$

$$0 < \Re(\sigma_{0t}) < 1, \tag{2.11}$$

$$\theta_0 \pm \theta_t \pm \sigma_{0t}, \qquad \theta_\infty \pm \theta_1 \pm \sigma_{0t} \notin 2\mathbb{Z}, \tag{2.12}$$

and we call these *generic conditions*. In our application the non-resonant condition (2.10) will be adhered to; however, (2.12) can be relaxed in a meaningful way and we conjecture that the results of Jimbo still hold under suitable limiting procedures where the left-hand sides $\to 2\mathbb{Z}$.

When $\sigma_{0t} \neq 0$, the parametrization of the monodromy matrices employed by Jimbo is

$$M_\infty = \begin{pmatrix} \mathrm{e}^{\pi\mathrm{i}\theta_\infty} & 0 \\ 0 & \mathrm{e}^{-\pi\mathrm{i}\theta_\infty} \end{pmatrix}, \tag{2.13}$$

$$M_1 = \frac{1}{\mathrm{i}\sin\pi\theta_\infty} \times \begin{pmatrix} \cos\pi\sigma - \mathrm{e}^{-\pi\mathrm{i}\theta_\infty}\cos\pi\theta_1 & -2r\,\mathrm{e}^{-\pi\mathrm{i}\theta_\infty}\sin\frac{\pi}{2}(\theta_\infty + \theta_1 + \sigma)\sin\frac{\pi}{2}(\theta_\infty + \theta_1 - \sigma) \\ 2r^{-1}\mathrm{e}^{\pi\mathrm{i}\theta_\infty}\sin\frac{\pi}{2}(\theta_\infty - \theta_1 + \sigma)\sin\frac{\pi}{2}(\theta_\infty - \theta_1 - \sigma) & -\cos\pi\sigma + \mathrm{e}^{\pi\mathrm{i}\theta_\infty}\cos\pi\theta_1 \end{pmatrix}, \tag{2.14}$$

$$DM_tD^{-1} = \frac{1}{\mathrm{i}\sin\pi\sigma} \times \begin{pmatrix} \mathrm{e}^{\pi\mathrm{i}\sigma}\cos\pi\theta_t - \cos\pi\theta_0 & -2s\mathrm{e}^{\pi\mathrm{i}\sigma}\sin\frac{\pi}{2}(\theta_0+\theta_t-\sigma)\sin\frac{\pi}{2}(\theta_0-\theta_t+\sigma) \\ 2s^{-1}\mathrm{e}^{-\pi\mathrm{i}\sigma}\sin\frac{\pi}{2}(\theta_0+\theta_t+\sigma)\sin\frac{\pi}{2}(\theta_0-\theta_t-\sigma) & -\mathrm{e}^{-\pi\mathrm{i}\sigma}\cos\pi\theta_t + \cos\pi\theta_0 \end{pmatrix}, \tag{2.15}$$

$$DM_0D^{-1} = \frac{1}{\mathrm{i}\sin\pi\sigma} \times \begin{pmatrix} \mathrm{e}^{\pi\mathrm{i}\sigma}\cos\pi\theta_0 - \cos\pi\theta_t & 2s\sin\frac{\pi}{2}(\theta_0+\theta_t-\sigma)\sin\frac{\pi}{2}(\theta_0-\theta_t+\sigma) \\ -2s^{-1}\sin\frac{\pi}{2}(\theta_0-\theta_t-\sigma)\sin\frac{\pi}{2}(\theta_0+\theta_t+\sigma) & -\mathrm{e}^{-\pi\mathrm{i}\sigma}\cos\pi\theta_0 + \cos\pi\theta_t \end{pmatrix}, \tag{2.16}$$

where

$$D = \begin{pmatrix} \sin\frac{\pi}{2}(\theta_\infty-\theta_1-\sigma) & r\sin\frac{\pi}{2}(\theta_\infty+\theta_1+\sigma) \\ r^{-1}\sin\frac{\pi}{2}(\theta_\infty-\theta_1+\sigma) & \sin\frac{\pi}{2}(\theta_\infty+\theta_1-\sigma) \end{pmatrix}, \tag{2.17}$$

with the short-hand notation $\sigma = \sigma_{0t}$, $s = s_{0t}$. The quantity s_{0t} together with σ_{0t} defines a unique solution to the P_{VI} σ-form (2.5). The other parameter r is an arbitrary nonzero complex constant and is a free parameter in the monodromy matrices but does not appear in the local expansions.

A key formula is the following connection relation which relates s_{0t}, σ_{0t} to σ_{t1} and σ_{01}.

Lemma 2.1 [5, 18]**.** *Under the above generic conditions (2.10), (2.11), (2.12) and notations*

$$\begin{aligned} 4s^{\pm1}\sin\frac{\pi}{2}(\theta_0+\theta_t\mp\sigma_{0t})\sin\frac{\pi}{2}(\theta_0-\theta_t\pm\sigma_{0t}) \\ \times\sin\frac{\pi}{2}(\theta_\infty+\theta_1\mp\sigma_{0t})\sin\frac{\pi}{2}(\theta_\infty-\theta_1\pm\sigma_{0t}) \\ = \mathrm{e}^{\pm\pi\mathrm{i}\sigma_{0t}}(\pm\mathrm{i}\sin\pi\sigma_{0t}\cos\pi\sigma_{t1} - \cos\pi\theta_t\cos\pi\theta_\infty - \cos\pi\theta_0\cos\pi\theta_1) \\ \pm\mathrm{i}\sin\pi\sigma_{0t}\cos\pi\sigma_{01} + \cos\pi\theta_t\cos\pi\theta_1 + \cos\pi\theta_\infty\cos\pi\theta_0. \end{aligned} \tag{2.18}$$

A consequence of this relation is a constraint on the monodromy invariants which is an algebraic variety defining a sub-manifold, the monodromy manifold, of $\mathbb{C}^3$.

Lemma 2.2 [18]**.** *The monodromy manifold for* P_{VI} *is given by*

$$\begin{aligned} \mathfrak{M}(p_{0t}, p_{t1}, p_{01}) = {} & p_{0t}p_{t1}p_{01} + p_{0t}^2 + p_{t1}^2 + p_{01}^2 \\ & - (p_0p_t + p_1p_\infty)p_{0t} - (p_tp_1 + p_0p_\infty)p_{t1} - (p_0p_1 + p_tp_\infty)p_{01} \\ & + p_0^2 + p_t^2 + p_1^2 + p_\infty^2 + p_0p_tp_1p_\infty - 4 = 0. \end{aligned} \tag{2.19}$$

Theorem 2.1 [18]**.** *Under conditions (2.10)–(2.12), we have the asymptotic expansion of the* τ*-function as* $t \to 0$ *in the domain* $\{t \in \mathbb{C} | 0 < |t| < \varepsilon, |\arg(t)| < \phi\}$ *for all* $\varepsilon > 0$ *and any* $\phi > 0$*:*

$$\begin{aligned} \tau(t) \sim \mathrm{const}\cdot t^{(\sigma^2-\theta_0^2-\theta_t^2)/4}\Bigg\{ & 1 + \frac{(\theta_0^2-\theta_t^2-\sigma^2)(\theta_\infty^2-\theta_1^2-\sigma^2)}{8\sigma^2}t \\ & - \hat{s}\frac{[\theta_0^2-(\theta_t-\sigma)^2][\theta_\infty^2-(\theta_1-\sigma)^2]}{16\sigma^2(1+\sigma)^2}t^{1+\sigma} \\ & - \hat{s}^{-1}\frac{[\theta_0^2-(\theta_t+\sigma)^2][\theta_\infty^2-(\theta_1+\sigma)^2]}{16\sigma^2(1-\sigma)^2}t^{1-\sigma} + \mathrm{O}(|t|^{2(1-\Re(\sigma))})\Bigg\}, \end{aligned} \tag{2.20}$$

where $\sigma \neq 0$ *and* $\hat{s}$ *are related to* s *through*

$$\hat{s} = s\frac{\Gamma^2(1-\sigma)\Gamma\left(1+\frac{1}{2}(\theta_0+\theta_t+\sigma)\right)\Gamma\left(1+\frac{1}{2}(-\theta_0+\theta_t+\sigma)\right)}{\Gamma^2(1+\sigma)\Gamma\left(1+\frac{1}{2}(\theta_0+\theta_t-\sigma)\right)\Gamma\left(1+\frac{1}{2}(-\theta_0+\theta_t-\sigma)\right)} \times \frac{\Gamma\left(1+\frac{1}{2}(\theta_\infty+\theta_1+\sigma)\right)\Gamma\left(1+\frac{1}{2}(-\theta_\infty+\theta_1+\sigma)\right)}{\Gamma\left(1+\frac{1}{2}(\theta_\infty+\theta_1-\sigma)\right)\Gamma\left(1+\frac{1}{2}(-\theta_\infty+\theta_1-\sigma)\right)}, \tag{2.21}$$

and we employ the short-hand notation $s = s_{0t}$, $\hat{s} = \hat{s}_{0t}$ *and* $\sigma = \sigma_{0t}$. *The monodromy data defining a unique solution to the sixth Painlevé system are* $\{\sigma_{0t}, s_{0t}\}$.

3. The spectrum singularity ensemble

A fundamental ensemble in random matrix theory is the ensemble of finite rank (rank $= N$) random unitary matrices—the *Dyson circular unitary ensemble (CUE)*. Consider a member of this ensemble U with eigenvalues $z_1 = \mathrm{e}^{\mathrm{i}\theta_l}, \ldots, z_N = \mathrm{e}^{\mathrm{i}\theta_N}$. Then the eigenvalue probability density function for the ensemble is the Haar measure for $U(N)$:

$$p_N(\theta_1, \ldots, \theta_N) := \frac{1}{(2\pi)^N N!} \prod_{1\leqslant j<k\leqslant N} |z_j - z_k|^2. \tag{3.1}$$

A generalization of this ensemble, referred to as the *spectrum singularity ensemble (SSE)* [15], has an eigenvalue probability density function containing additional algebraic singularities at $z = 0, -1, -1/t$, where $t = \mathrm{e}^{\mathrm{i}\phi}, \phi \in [0, 2\pi)$. In the log-gas picture of the CUE the eigenvalues of a random unitary matrix are mobile unit charges subject to a pairwise mutual logarithmic repulsion and in the SSE these charges are subject to additional external fields—the logarithmic electrostatic potential of impurity charges located at the ends of the sector, at $z = -1, -1/t$ with charges ω_1, μ respectively, and an external electric field of strength ω_2. The generalized spectral average we wish to investigate is the partition function of such an electrostatic system, and is given by the N-dimensional integral [15]

$$\mathcal{A}_N(t; \omega_1, \omega_2, \mu; \xi^*) := \frac{1}{N!}\left(\int_{-\pi}^{\pi} - \xi^* \int_{\pi-\phi}^{\pi}\right)\frac{\mathrm{d}\theta_1}{2\pi} \cdots \left(\int_{-\pi}^{\pi} - \xi^* \int_{\pi-\phi}^{\pi}\right)\frac{\mathrm{d}\theta_N}{2\pi} \times \prod_{l=1}^{N} z_l^{-\mathrm{i}\omega_2}|1+z_l|^{2\omega_1}|1+tz_l|^{2\mu} \prod_{1\leqslant j<k\leqslant N} |z_j - z_k|^2, \tag{3.2}$$

where $\xi^* \in \mathbb{C}$ and the parameters $\omega_1, \omega_2, \mu \in \mathbb{C}, \omega = \omega_1 + \mathrm{i}\omega_2$, are restricted with $\Re(2\omega_1), \Re(2\mu) > -1, N \in \mathbb{Z}_{\geqslant 0}$. The independent variable t, whilst defined on the unit circle $|t| = 1$, can be analytically continued into the cut complex t-plane.

It was shown in [15] that the average $\mathcal{A}_N(t)$ is the $N \in \mathbb{Z}_{\geqslant 0}$th member of a sequence of classical τ-functions of the sixth Painlevé system. Thus this average is characterized by solutions of the nonlinear differential equation (2.5) subject to the boundary conditions given in [17], which were derived following an idea introduced in [16].

Theorem 3.1 [17]**.** *For generic values of the parameters* $\mu, \omega, \bar{\omega}$, *subject to* $2\mu + 2\omega_1 \notin \mathbb{Z}$, $\Re(2\mu + 2\omega_1) > 0$, *the spectral average* $\mathcal{A}_N$ *has the following expansion about* $t = 1$*:*

$$
\begin{aligned}
\mathcal{A}_N(t) = {} & \prod_{k=0}^{N-1} \frac{k!\Gamma(2\mu+2\omega_1+k+1)}{\Gamma(1+k+\mu+\omega)\Gamma(1+k+\mu+\bar{\omega})} \left\{ 1 + \frac{N\mu(\bar{\omega}-\omega)}{2\mu+2\omega_1}(1-t) + \mathrm{O}((1-t)^2) \right. \\
& + \frac{(-1)^{N+1}}{\sin\pi(2\mu+2\omega_1)} \left(\xi^* \frac{\mathrm{e}^{-\pi\mathrm{i}(\mu-\bar{\omega})}}{2\mathrm{i}} + \frac{\sin\pi 2\mu \sin\pi(\mu+\omega)}{\sin\pi(2\mu+2\omega_1)} \right) \\
& \times \frac{\Gamma(1+2\mu)\Gamma(1+2\omega_1)\Gamma(1+\mu+\omega)\Gamma(1+\mu+\bar{\omega})}{\Gamma^2(2\mu+2\omega_1+2)\Gamma(2\mu+2\omega_1+1)\Gamma(N)\Gamma(-N-2\mu-2\omega_1)} (1-t)^{1+2\mu+2\omega_1} \\
& \left. \times (1+\mathrm{O}(1-t)) + \mathrm{O}((1-t)^{2+4\mu+4\omega_1}) \right\}. \qquad (3.3)
\end{aligned}
$$

The precise relationship between the spectrum singularity average $\mathcal{A}_N(t)$ and the isomonodromy theory of the sixth Painlevé system was first given in [17]. However because we intend to apply the Kitaev theory [20], which treats the coalescence of the regular singularities $0, t$ of the $\mathrm{P_{VI}}$ system under the transition limit $t \to 0$, and the natural bulk scaling of $A_N(t)$ is through the limit $t \to 1$, the monodromy data of our application will have to be recast in a suitable form.

Theorem 3.2 [17]**.** *The spectrum singularity ensemble can be defined by the following monodromy data. The formal monodromy exponents are*

$$\theta_0 = N + 2\mu, \qquad \theta_t = -N - 2\omega_1, \qquad \theta_1 = -\mu - \omega, \qquad \theta_\infty = \mu + \bar{\omega}, \qquad (3.4)$$

with the necessary classical condition $\theta_0 - \theta_t + \theta_1 - \theta_\infty \in 2\mathbb{Z}$*. The relevant monodromy invariant is*

$$\sigma_{0t} = 2\mu + 2\omega_1. \qquad (3.5)$$

In this case, the associated monodromy coefficient is given by $s_{0t} = \epsilon\pi\hat{s}/2$ *under the limiting process* $\epsilon := \sigma_{0t} + \theta_1 - \theta_\infty \to 0$*, along with*

$$\hat{s}\sin\pi 2\omega_1 \sin\pi(\mu+\omega) = \frac{\sin\pi 2\mu \sin\pi(\mu+\omega)}{\sin\pi(2\mu+2\omega_1)} + \xi^* \frac{\mathrm{e}^{-\pi\mathrm{i}(\mu-\bar{\omega})}}{2\mathrm{i}}. \qquad (3.6)$$

All monodromy matrices are upper triangular

$$M_0 = \begin{pmatrix} \mathrm{e}^{-\pi\mathrm{i}\theta_0} & m_0 \\ 0 & \mathrm{e}^{\pi\mathrm{i}\theta_0} \end{pmatrix}, \qquad (3.7)$$

$$M_t = \begin{pmatrix} \mathrm{e}^{\pi\mathrm{i}\theta_t} & m_t \\ 0 & \mathrm{e}^{-\pi\mathrm{i}\theta_t} \end{pmatrix}, \qquad (3.8)$$

$$M_1 = \begin{pmatrix} \mathrm{e}^{-\pi\mathrm{i}\theta_1} & m_1 \\ 0 & \mathrm{e}^{\pi\mathrm{i}\theta_1} \end{pmatrix}, \qquad (3.9)$$

with upper triangular elements of the form

$$m_0 = (-1)^N 2\mathrm{i}r \frac{\sin\pi 2\mu}{\sin^2\pi(2\mu+2\omega_1)} \left\{ -\frac{r}{\hat{s}} \sin\pi(\mu+\bar{\omega}) + \sin\pi(2\mu+2\omega_1)\sin\pi(\mu+\omega) \right\}, \qquad (3.10)$$

$$
\begin{aligned}
m_t = (-1)^N 2\mathrm{i}r \frac{1}{\sin^2\pi(2\mu+2\omega_1)} & \left\{ \frac{r}{\hat{s}} \mathrm{e}^{-\pi\mathrm{i}(2\mu+2\omega_1)} \sin\pi 2\mu \sin\pi(\mu+\bar{\omega}) \right. \\
& \left. + \sin\pi(2\mu+2\omega_1)\sin\pi 2\omega_1 \sin\pi(\mu+\omega) \right\}, \qquad (3.11)
\end{aligned}
$$

$$m_1 = -2\mathrm{i}r\,\mathrm{e}^{-\pi\mathrm{i}(\mu+\bar{\omega})}\sin\pi(\mu+\omega), \qquad (3.12)$$

where r *is a nonzero arbitrary constant.*

Proof. In [17], it was noted that the monodromy exponents could be given by any one of the three sets

$$\{\theta_0, \theta_t, \theta_1, \theta_\infty\} = \begin{cases} N + 2\mu,\, N + 2\omega_1,\, \mu + \omega,\, \mu + \bar{\omega} \\ N,\, N + 2\mu + 2\omega_1,\, \mu - \omega,\, \mu - \bar{\omega} \\ N + \mu + \omega,\, N + \mu + \bar{\omega},\, -2\mu,\, 2\omega_1, \end{cases} \tag{3.13}$$

modulo permutations of the exponents, an even number of sign reversals and subject to some constraints which we elaborate later on. It was found that the first set led to upper triangular monodromy matrices, the second to full monodromy matrices except for one which was a multiple of the identity, and the third to lower triangular matrices. The second set strictly violates the condition of non-resonant monodromy exponents, and the third set is essentially equivalent to the first and therefore we choose an example from the first set. These constraints, when applied to the first set, imply that our choices are now

$$\{\theta_0, \theta_t, \theta_1, \theta_\infty\} = \begin{cases} \pm(\mu + \omega),\, \pm(N + 2\mu),\, \pm(N + 2\omega_1),\, \pm(\mu + \bar{\omega}) \\ \pm(\mu + \bar{\omega}),\, \pm(N + 2\omega_1),\, \pm(N + 2\mu),\, \pm(\mu + \omega) \\ \pm(\mu + \omega),\, \pm(N + 2\mu),\, \pm(N + 2\omega_1),\, \pm(\mu + \bar{\omega}) \\ \pm(\mu + \bar{\omega}),\, \pm(N + 2\omega_1),\, \pm(N + 2\mu),\, \pm(\mu + \omega), \end{cases} \tag{3.14}$$

with an even number of sign reversals. The monodromy invariants were given as $\sigma_{0t} = N - \mu + \bar{\omega}, \sigma_{t1} = 2\mu + 2\omega_1, \sigma_{01} = N - \mu + \omega$.

Next we employ the linear fractional transformation of our original system $t \mapsto 1 - t$, $\theta_0 \leftrightarrow \theta_1, \sigma_{0t} \mapsto \sigma_{t1}, \sigma_{t1} \mapsto \sigma_{0t}$, and it is this new system to which we shall refer to henceforth. We take one choice from the above (3.14) which is applicable to the theory given in [20] and this is (3.4) and (3.5). Having made this choice, we observe that in order for the Jimbo parametrization given in (2.20) and (2.21) to be consistent with (3.3) under the map $t \mapsto 1 - t$, s_{0t} must vanish in the manner given in theorem (3.2). The resulting finite coefficient $\hat{s}$ is then given by formula (3.6). It can be also verified that the overall prefactors and the leading order analytic terms in both expansions agree precisely with the choices we have made. The monodromy matrices are then given by (3.7)–(3.12) using the parametrization in (2.14), (2.15), (2.16), (2.17). The off-diagonal elements satisfy the relation $\mathrm{e}^{-\pi\mathrm{i}\theta_0}m_0 + \mathrm{e}^{-\pi\mathrm{i}\theta_t}m_t + \mathrm{e}^{-\pi\mathrm{i}(\theta_0+\theta_t-\theta_\infty)}m_1 = 0$, as required by the cyclic identity (2.9). □

4. Isomonodromy deformation formulation for $\mathrm{P_V}$

The work of Jimbo [18] has a slightly different formulation of the isomonodromy problem for $\mathrm{P_V}$ from that of Kitaev and collaborators [1, 2, 20], and it is the latter form that we adopt. We formulate the $\mathrm{P_V}$ isomonodromic system as the Lax pair of linear 2×2 matrix ODEs for $\Psi(\lambda; t)$ with two regular singular points at $\nu = 0, 1$ and an irregular one at ∞ with the Poincare rank unity

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Psi = \left(\frac{t}{2}\sigma_3 + \frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1}\right)\Psi, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Psi = \left\{\frac{\lambda}{2}\sigma_3 + \frac{1}{t}\left(\frac{\theta_\infty}{2}\sigma_3 + A_0 + A_1\right)\right\}\Psi. \tag{4.1}$$

We require that the residue matrices satisfy $\mathrm{diag}(A_0 + A_1) = -\frac{\theta_\infty}{2}\sigma_3, \theta_\infty \in \mathbb{C}\backslash\mathbb{Z}$, and the constraints $\mathrm{tr}\, A_\nu = 0, \det A_\nu = -\frac{1}{4}\theta_\nu^2, \nu \in \{0, 1\}$. Again we assume that there exist invertible matrices $R_\nu \in SL(2, \mathbb{C})$ such that $R_\nu^{-1} A_\nu R_\nu = \frac{1}{2}\theta_\nu\sigma_3, \theta_\nu \in \mathbb{C}\backslash\mathbb{Z}, \nu \in \{0, 1\}$.

Within the works of Kitaev and collaborators there are two different conventions for the monodromy data employed, and we will make the distinction between the two by using a carét for those of [20] as opposed to those of [1, 2]. In the neighbourhood of the irregular singularity, the canonical solutions $\Psi^k(\lambda)$ can be expanded as

$$\Psi^k(\lambda) \sim \left(I + \sum_{m=1}^{\infty} \Psi^k_{m\infty}\lambda^{-m}\right) \exp\left\{\left(\frac{t}{2}\lambda - \frac{1}{2}\theta_\infty \log\lambda\right)\sigma_3\right\} \tag{4.2}$$

in the sectors $-3\pi/2+\pi k < \arg\lambda t < \pi/2+\pi k$, $k = 1, 2$. The Stokes matrices S_k are defined as $\Psi^{k+1}(\lambda) = \Psi^k(\lambda)\hat{S}_k$, which have the structure

$$\hat{S}_{2l} = \begin{pmatrix} 1 & 0 \\ \hat{s}_{2l} & 1 \end{pmatrix}, \qquad \hat{S}_{2l+1} = \begin{pmatrix} 1 & \hat{s}_{2l+1} \\ 0 & 1 \end{pmatrix}, \tag{4.3}$$

where $\hat{s}_k$ are the Stokes multipliers. The monodromy matrix $\hat{M}_{k\infty}$ is given by $\hat{M}_{k\infty} = \hat{S}_k\hat{S}_{k+1}\,\mathrm{e}^{\pi\mathrm{i}\theta_\infty\sigma_3}$, $C_\infty = I$, and we set $\hat{M}_\infty = \hat{M}_{0\infty}$. The monodromy matrices at the regular singularities are defined in the same way as in (2.8). The cyclic relation in this case is

$$\hat{M}_0\hat{M}_1\hat{M}_\infty = I, \tag{4.4}$$

reversed in order from the usual convention because of the nature of the limiting transition we are to consider later. The monodromy data preserved here are $\mathcal{M}_{\mathrm{V}} := \{\theta_\nu, C_\nu, \hat{M}_\nu, \hat{s}_1, \hat{s}_2\}$. If we define the monodromy invariant $2\cos\pi\sigma = \mathrm{Tr}(\hat{M}_0\hat{M}_1)$, then the following constraint, analogous to (2.19), applies:

$$\hat{s}_0\hat{s}_1\,\mathrm{e}^{-\pi\mathrm{i}\theta_\infty} = 4\sin\frac{\pi}{2}(\theta_\infty+\sigma)\sin\frac{\pi}{2}(\theta_\infty-\sigma). \tag{4.5}$$

In contrast, the cyclic relation adopted in [1, 2] is

$$M_\infty M_1 M_0 = I, \tag{4.6}$$

where

$$M_\infty \equiv M_{2\infty} = S_2\,\mathrm{e}^{\pi\mathrm{i}\theta_\infty\sigma_3}S_1. \tag{4.7}$$

The inter-relationship between the two sets of monodromy data is given by

$$\hat{S}_0 = S_1, \qquad \hat{S}_1 = S_2 \tag{4.8}$$

$$\hat{M}_0 = S_1M_1M_0M_1^{-1}S_1^{-1}, \qquad \hat{M}_1 = S_1M_1S_1^{-1}, \qquad \hat{M}_\infty = S_1M_\infty S_1^{-1}. \tag{4.9}$$

The σ-function in the Jimbo formulation of the $\mathrm{P_V}$ linear system is equivalent to that of Andreev and Kitaev [2] and is defined as

$$\zeta(t) = t\frac{\mathrm{d}}{\mathrm{d}t}\log\tau + \frac{1}{2}(\theta_0+\theta_\infty)t + \frac{1}{4}\left[(\theta_0+\theta_\infty)^2 - \theta_1^2\right], \tag{4.10}$$

which satisfies the second-order second-degree differential equation

$$\begin{aligned}\left(t\frac{\mathrm{d}^2}{\mathrm{d}t^2}\zeta\right)^2 &= \left[\zeta - t\frac{\mathrm{d}}{\mathrm{d}t}\zeta + 2\left(\frac{\mathrm{d}}{\mathrm{d}t}\zeta\right)^2 - (2\theta_0+\theta_\infty)\frac{\mathrm{d}}{\mathrm{d}t}\zeta\right]^2 \\ &\quad - 4\frac{\mathrm{d}}{\mathrm{d}t}\zeta\left(\frac{\mathrm{d}}{\mathrm{d}t}\zeta - \theta_0\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\zeta - \frac{1}{2}(\theta_0-\theta_1+\theta_\infty)\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\zeta - \frac{1}{2}(\theta_0+\theta_1+\theta_\infty)\right).\end{aligned} \tag{4.11}$$

The asymptotic expansion of the τ-function is given by the following theorem.

Theorem 4.1 [18]. *Under the conditions $\theta_0, \theta_1 \notin \mathbb{Z}$, $0 < \Re(\sigma) < 1$ and $\theta_1 \pm \theta_0 \pm \sigma, \theta_\infty \pm \sigma \notin 2\mathbb{Z}$, then we have the asymptotic expansion of the τ-function as $t \to 0$:*

$$\tau(t) \sim \text{const} \cdot t^{(\sigma^2-\theta_\infty^2)/4}\left\{1 - \frac{\theta_\infty\left(\theta_1^2-\theta_0^2+\sigma^2\right)}{4\sigma^2}t - \hat{s}\frac{[\theta_\infty-\sigma]\left[\theta_0^2-(\theta_1-\sigma)^2\right]}{8\sigma^2(1+\sigma)^2}t^{1+\sigma}\right.$$
$$\left. - \hat{s}^{-1}\frac{[\theta_\infty+\sigma]\left[\theta_0^2-(\theta_1+\sigma)^2\right]}{8\sigma^2(1-\sigma)^2}t^{1-\sigma} + \mathrm{O}(|t|^{2(1-\Re(\sigma))})\right\}, \tag{4.12}$$

where $\sigma \neq 0$ and $\hat{s}$ are related to s through

$$\hat{s} = s\frac{\Gamma^2(1-\sigma)\Gamma\left(1+\frac{1}{2}(\theta_1+\theta_0+\sigma)\right)\Gamma\left(1+\frac{1}{2}(\theta_1-\theta_0+\sigma)\right)}{\Gamma^2(1+\sigma)\Gamma\left(1+\frac{1}{2}(\theta_1+\theta_0-\sigma)\right)\Gamma\left(1+\frac{1}{2}(\theta_1-\theta_0-\sigma)\right)}\frac{\Gamma\left(1+\frac{1}{2}(\theta_\infty+\sigma)\right)}{\Gamma\left(1+\frac{1}{2}(\theta_\infty-\sigma)\right)}. \tag{4.13}$$

When $\sigma \neq \mathbb{Z}$ we have the parametrization of the $\mathrm{P_V}$ monodromy matrices employed in [18], and in theorem 6.2 of [2]

$$DM_0D^{-1} = \frac{1}{\mathrm{i}\sin\pi\sigma}$$
$$\times\begin{pmatrix} \mathrm{e}^{\pi\mathrm{i}\sigma}\cos\pi\theta_0 - \cos\pi\theta_1 & -2rs\mathrm{e}^{-\pi\mathrm{i}\sigma}\sin\frac{\pi}{2}(\theta_1-\theta_0-\sigma)\sin\frac{\pi}{2}(\theta_1+\theta_0-\sigma) \\ 2(rs)^{-1}\mathrm{e}^{\pi\mathrm{i}\sigma}\sin\frac{\pi}{2}(\theta_1-\theta_0+\sigma)\sin\frac{\pi}{2}(\theta_1+\theta_0+\sigma) & -\mathrm{e}^{-\pi\mathrm{i}\sigma}\cos\pi\theta_0 + \cos\pi\theta_1 \end{pmatrix}, \tag{4.14}$$

$$DM_1D^{-1} = \frac{1}{\mathrm{i}\sin\pi\sigma}$$
$$\times\begin{pmatrix} \mathrm{e}^{\pi\mathrm{i}\sigma}\cos\pi\theta_1 - \cos\pi\theta_0 & 2rs\sin\frac{\pi}{2}(\theta_1-\theta_0-\sigma)\sin\frac{\pi}{2}(\theta_0+\theta_1-\sigma) \\ -2(rs)^{-1}\sin\frac{\pi}{2}(\theta_1-\theta_0+\sigma)\sin\frac{\pi}{2}(\theta_0+\theta_1+\sigma) & -\mathrm{e}^{-\pi\mathrm{i}\sigma}\cos\pi\theta_1 + \cos\pi\theta_0 \end{pmatrix}, \tag{4.15}$$

where

$$D = \begin{pmatrix} \mathrm{e}^{-\pi\mathrm{i}\sigma/2} & r\sin\frac{\pi}{2}(\theta_\infty+\sigma) \\ r^{-1}\mathrm{e}^{\pi\mathrm{i}\sigma/2} & \sin\frac{\pi}{2}(\theta_\infty-\sigma) \end{pmatrix}. \tag{4.16}$$

The Stokes multipliers are given by the formulae

$$s_1 = -\frac{2\pi\mathrm{i}r^{-1}}{\Gamma\left(1-\frac{\sigma-\theta_\infty}{2}\right)\Gamma\left(\frac{\sigma+\theta_\infty}{2}\right)}, \qquad s_2 = -\mathrm{e}^{\pi\mathrm{i}\theta_\infty}\frac{2\pi\mathrm{i}r}{\Gamma\left(1-\frac{\sigma+\theta_\infty}{2}\right)\Gamma\left(\frac{\sigma-\theta_\infty}{2}\right)}. \tag{4.17}$$

The parameter r is again an arbitrary nonzero complex constant and does not appear in the expansion formulae.

5. The bulk scaling ensemble

Also studied in [15] was the scaling limit of the spectrum singularity ensemble in the neighbourhood of the singularity at $\phi = 0$ $(t = 1)$, to the bulk regime via the limit $N \to \infty$. It was shown there that the log derivative of the average $\mathcal{A}_N(t)$ converged to a solution of the Jimbo–Miwa–Okamoto σ-function for the fifth Painlevé equation. Thus the most general universality class of Hermitian random matrix ensembles in the bulk scaling limit was found to be a generic four parameter class—three arbitrary parameters, ω_1, ω_2, μ, appearing in the differential equation and one, ξ^*, in the boundary data. Thus this class is characterized by a transcendental solution to a generic fifth Painlevé equation.

The establishment of this result was based on a formal scaling limit of the $\mathrm{P_{VI}}$ second-order second-degree ordinary differential equation (2.5) to the corresponding $\mathrm{P_V}$ ODE (4.11). Defining the scaling variables

$$t := \mathrm{e}^{-x/N}, \qquad u(x;\omega_1,\omega_2,\mu;\xi^*) := x\frac{\mathrm{d}}{\mathrm{d}x}\lim_{N\to\infty}\log\mathcal{A}_N(t;\omega_1,\omega_2,\mu;\xi^*), \tag{5.1}$$

it was found that $u(x)$ is related to a solution of an alternative Jimbo–Miwa–Okamoto σ-form of the fifth Painlevé equation $h_{\mathrm{V}}(s)$ by

$$u(ix;\omega_1,\omega_2,\mu) = h_{\mathrm{V}}(x;\mathbf{v}) + \left(-\tfrac{1}{4}\omega - \tfrac{3}{4}\bar{\omega} + \mu\right)x + \tfrac{1}{8}(\omega-\bar{\omega})^2 - \mu(\omega+\bar{\omega}), \tag{5.2}$$

with the Okamoto parameters

$$(v_1,v_2,v_3,v_4) = \left(\omega_1 - \tfrac{1}{2}\mathrm{i}\omega_2, -\omega_1 - \tfrac{1}{2}\mathrm{i}\omega_2, \mu + \tfrac{1}{2}\mathrm{i}\omega_2, -\mu + \tfrac{1}{2}\mathrm{i}\omega_2\right). \tag{5.3}$$

The alternative Jimbo–Miwa–Okamoto σ-form of the fifth Painlevé equation (4.11) is

$$(xh_{\mathrm{V}}'')^2 - [h_{\mathrm{V}} - xh_{\mathrm{V}}' + 2(h_{\mathrm{V}}')^2]^2 + 4\prod_{k=1}^{4}(h_{\mathrm{V}}' + v_k) = 0, \tag{5.4}$$

where the constraint $v_1+v_2+v_3+v_4=0$ applies. The scaled spectral average is a τ-function for this system and related to $u(x)$ by

$$\mathcal{A}(x) = \exp\int_0^x \frac{\mathrm{d}y}{y}u(y;\omega_1,\omega_2,\mu). \tag{5.5}$$

We have the following expansion of $\mathcal{A}(x)$ about $x=0$ under the above formal bulk scaling limit.

Theorem 5.1. *Under the conditions* $\Re(2\mu), \Re(2\omega_1), \Re(\mu+\omega), \Re(\mu+\bar{\omega}) > -1, 0 \leqslant \Re(2\mu+2\omega_1) < 1$*, the spectral average* $\mathcal{A}_N(t;\omega_1,\omega_2,\mu;\xi^*)$ *has a bulk scaling limit with* $t \mapsto \exp(-x/N)$ *as* $N\to\infty$*,*

$$\prod_{k=0}^{N-1}\frac{\Gamma(1+k+\mu+\omega)\Gamma(1+k+\mu+\bar{\omega})}{k!\Gamma(2\mu+2\omega_1+k+1)}\mathcal{A}_N(t;\omega_1,\omega_2,\mu;\xi^*) \underset{N\to\infty}{\sim} \mathcal{A}(x), \tag{5.6}$$

and the scaled $\mathcal{A}$*-function has the following expansion as* $x\to 0$*:*

$$\begin{aligned}\mathcal{A}(x) = 1 &+ \frac{\mu(\bar{\omega}-\omega)}{2\mu+2\omega_1}x + \mathrm{O}(x^2) + \frac{1}{\pi}\left(\xi^*\frac{\mathrm{e}^{-\pi\mathrm{i}(\mu-\bar{\omega})}}{2\mathrm{i}} + \frac{\sin\pi 2\mu\sin\pi(\mu+\omega)}{\sin\pi(2\mu+2\omega_1)}\right)\\ &\times\frac{\Gamma(1+2\mu)\Gamma(1+2\omega_1)\Gamma(1+\mu+\omega)\Gamma(1+\mu+\bar{\omega})}{\Gamma^2(2\mu+2\omega_1+2)\Gamma(2\mu+2\omega_1+1)}x^{1+2\mu+2\omega_1}(1+\mathrm{O}(x))\\ &+\mathrm{O}(x^{2+4\mu+4\omega_1}).\end{aligned} \tag{5.7}$$

Proof. Expansion (3.3) is valid uniformly in N under the substitution $t\mapsto\exp(-x/N)$ and by taking the limit $N\to\infty$, and employing the asymptotic formula for the ratio of gamma functions we arrive at (5.7). □

Comparison of this result with the general theorem 4.1 of Jimbo leads to the following conclusions.

Theorem 5.2. *The* $\mathrm{P_V}$ *monodromy parameters for the bulk scaled spectrum singularity ensemble are*

$$\theta_0 = \mu+\bar{\omega}, \qquad \theta_1 = -\mu-\omega, \qquad \theta_\infty = 2\mu - 2\omega_1, \tag{5.8}$$

with the invariant $\sigma = 2\mu+2\omega_1$*. The monodromy coefficient is*

$$s = -\frac{\sin\pi 2\mu}{\sin\pi 2\omega_1} - \frac{\sin\pi(2\mu+2\omega_1)}{\sin\pi 2\omega_1\sin\pi(\mu+\omega)}\xi^*\frac{\mathrm{e}^{-\pi\mathrm{i}(\mu-\bar{\omega})}}{2\mathrm{i}}. \tag{5.9}$$

Proof. The identification of (5.4) with the parameter set (5.3) with (4.11) leads to the following solution sets for the $\mathrm{P_V}$ formal monodromy exponents:

$$\left\{\tfrac{1}{2}\theta_0+\tfrac{1}{4}\theta_\infty, -\tfrac{1}{2}\theta_0+\tfrac{1}{4}\theta_\infty, \tfrac{1}{2}\theta_1-\tfrac{1}{4}\theta_\infty, -\tfrac{1}{2}\theta_1-\tfrac{1}{4}\theta_\infty\right\}$$
$$= \left\{\mu-\tfrac{1}{2}\mathrm{i}\omega_2, -\mu-\tfrac{1}{2}\mathrm{i}\omega_2, \omega-\tfrac{1}{2}\mathrm{i}\omega_2, -\bar{\omega}-\tfrac{1}{2}\mathrm{i}\omega_2\right\}, \tag{5.10}$$

modulo permutations. One choice, consistent with subsequent calculations, is given in (5.8). Using this choice and comparing the two asymptotic expansions (5.7) and (4.12) through their relationship

$$\mathcal{A}(x) = C\,\mathrm{e}^{\frac{1}{4}(\theta_\infty-\omega+\bar{\omega})x}x^{\frac{1}{4}[(\theta_0+\theta_\infty)^2-\theta_1^2]-\frac{1}{2}(\theta_0+\frac{1}{2}\theta_\infty)^2-2\mu\omega_1+\frac{1}{8}(\omega-\bar{\omega})^2}\tau(x), \tag{5.11}$$

we see that $\sigma^2=(2\mu+2\omega_1)^2$ from consideration of the algebraic prefactor. The coefficients of the analytic terms then agree upon using this value for the exponent. Finally, the coefficient of the non-analytic $x^{1-\sigma}$ vanishes because $\theta_0-\theta_1-\sigma=0$ and (5.9) follows from a matching of the remaining term. □

6. The limit transition $\mathrm{P_{VI}}$ to $\mathrm{P_V}$

The limit transition from $\mathrm{P_{VI}}$ to $\mathrm{P_V}$ is from one monodromy preserving system to another, and it is known [6] that there exists no continuous transition between such systems which is itself monodromy preserving. However, there are discrete transitions which are monodromy preserving and one example is a sequence of even parity Schlesinger transformations of the $\mathrm{P_{VI}}$ system, i.e. $\theta_\nu\mapsto\theta_\nu+2n, n\in\mathbb{Z}^+$. In a sense the $\mathrm{P_V}$ system arises then as a fixed point of this map, although the deformation variable also must scale in an appropriate way. This approach was fundamental to the study of Kitaev [20] who considered an example which is relevant to the present work. By way of comparison with our theorem 5.1, let us now consider the formal scaling undertaken in limit II of $\mathrm{P_{VI}}$ as given in [20]. In this limit, the $\mathrm{P_{VI}}$ formal exponents of monodromy are related to the $\mathrm{P_V}$ exponents by

$$\theta_{t\mathrm{VI}} = \theta_6-2n =: -\frac{1}{\epsilon}, \tag{6.1}$$

$$\theta_{0\mathrm{VI}} = \theta_{\infty\mathrm{V}}-\theta_6+2n = \theta_{\infty\mathrm{V}}+\frac{1}{\epsilon}, \tag{6.2}$$

$$\theta_{1\mathrm{VI}} = \theta_{1\mathrm{V}}, \tag{6.3}$$

$$\theta_{\infty\mathrm{VI}} = \theta_{0\mathrm{V}}, \tag{6.4}$$

and the limit $n\to\infty$ or $\epsilon\to0$ is taken. The quantity θ_6 is an additional constant which will be fixed in our application. The two regular singularities $\nu=0,t$ coalesce through $t_{\mathrm{VI}}=\epsilon t_{\mathrm{V}}=O(\epsilon)$, and the transition from the linear $\mathrm{P_{VI}}$ system to the $\mathrm{P_V}$ system takes place according to the scheme displayed in figure 2.

At the level of the $\mathrm{P_{VI}}$ isomonodromic system, the following scaling takes place as $n\to\infty, \epsilon\to0$:

$$\lambda_{\mathrm{VI}} = \frac{1}{\lambda_{\mathrm{V}}}, \tag{6.5}$$

$$t_{\mathrm{VI}} = \epsilon t_{\mathrm{V}}, \tag{6.6}$$

$$\lim_{\epsilon\to0} R_{t\mathrm{VI}}^{-1}\Psi_{\mathrm{VI}}(t_{\mathrm{VI}},\lambda_{\mathrm{VI}}) = \Psi_{\mathrm{V}}(t_{\mathrm{V}},\lambda_{\mathrm{V}}), \tag{6.7}$$

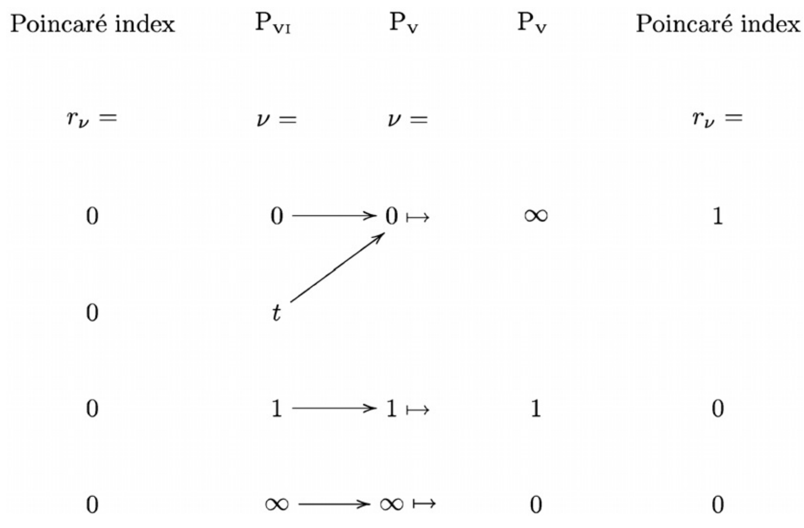

Figure 2. Coalescence of the two regular singularities $0, t$ of the P_{VI} system into the irregular singularity at ∞ of a P_V system according to the limit transition II of [20].

$$R_{tVI}^{-1}\frac{1}{2}\theta_{\infty VI}\sigma_3 R_{tVI} = A_{0V} + O(\epsilon), \tag{6.8}$$

$$R_{tVI}^{-1}A_{1VI}R_{tVI} = A_{1V} + O(\epsilon), \tag{6.9}$$

$$-R_{tVI}^{-1}\frac{d}{dt_V}R_{tVI} = \frac{1}{t_V}\left(\frac{1}{2}\theta_{\infty V}\sigma_3 + A_{0V} + A_{1V}\right) + O(\epsilon). \tag{6.10}$$

Consequently theorem 2 of Kitaev [20] allows us to compute the P_V monodromy data under the limit transition II described above.

Theorem 6.1 [20]. *Let $\mu_{VI} \in \mathcal{M}_{VI}(\theta_{\infty V} - \theta_6, \theta_{1VI}, \theta_6, \theta_{\infty VI})$ and suppose the following conditions hold:*

(1) (6.1), (6.2) and $\theta_{\nu VI} \in \mathbb{C}\backslash\mathbb{Z}$;
(2) $(M_{tVI} - e^{-\pi i\theta_6})(M_{0VI} - e^{-\pi i(\theta_{\infty V}-\theta_6)}) \neq 0$;
(3) $\alpha, \beta, l \in \mathbb{C}\backslash\mathbb{Z}$, where

$$T = \mathrm{Tr}(M_{tVI} - e^{\pi i\theta_6})(M_{0VI} - e^{-\pi i(\theta_{\infty V}-\theta_6)}), \tag{6.11}$$

$$l = \frac{1}{\pi i}\log\left(\cos\pi\theta_{\infty V} + \frac{1}{2}T \pm \sqrt{\left(\cos\pi\theta_{\infty V} + \frac{1}{2}T\right)^2 - 1}\right), \tag{6.12}$$

$$\alpha = -\frac{1}{2}(\theta_{\infty V} - l), \qquad \beta = -\frac{1}{2}(\theta_{\infty V} + l); \tag{6.13}$$

(4) the inverse monodromy problem for (2.1) is solvable for all pairs (μ_{VI}, t_{VI}) such that $t_{VI} = \epsilon t_V > 0$, and
(5) it is possible to construct the sequence $A_{\nu VI}^{2n}(\epsilon t_V)$, $n \in \mathbb{Z}^+$.

Then the data $\mu_{\mathrm{V}} \in \mathcal{M}_{\mathrm{V}}(\theta_{0\mathrm{V}}, \theta_{1\mathrm{V}}, \theta_{\infty\mathrm{V}})$ are given by the formulae

$$\theta_{0\mathrm{V}} = \theta_{\infty\mathrm{VI}}, \qquad \theta_{1\mathrm{V}} = \theta_{1\mathrm{VI}}, \tag{6.14}$$

$$\hat{S}_0 = \begin{pmatrix} 1 & 0 \\ \frac{2\pi\mathrm{i}}{\Gamma(1-\alpha)\Gamma(1-\beta)} & 1 \end{pmatrix}, \qquad \hat{S}_1 = \begin{pmatrix} 1 & -\frac{2\pi\mathrm{i}}{\Gamma(\alpha)\Gamma(\beta)}\,\mathrm{e}^{\pi\mathrm{i}\theta_{\infty\mathrm{V}}} \\ 0 & 1 \end{pmatrix}, \tag{6.15}$$

$$\hat{M}_{0\mathrm{V}} = K M_{\infty\mathrm{VI}} K^{-1}, \qquad \hat{M}_{1\mathrm{V}} = K M_{1\mathrm{VI}} K^{-1}, \tag{6.16}$$

where K is the unique (up to a sign) solution of the system

$$K M_{t\mathrm{VI}} K^{-1} = \hat{S}_0\,\mathrm{e}^{\pi\mathrm{i}\theta_6\sigma_3}, \qquad K M_{0\mathrm{VI}} K^{-1} = \mathrm{e}^{-\pi\mathrm{i}\theta_6\sigma_3}\,\hat{S}_1\,\mathrm{e}^{\pi\mathrm{i}\theta_{\infty\mathrm{V}}\sigma_3}. \tag{6.17}$$

Applying the above theorem to the spectrum singularity ensemble we have the following result.

Corollary 6.1. *In the limit transition of theorem (6.1), we find that the Painlevé V parameters of the spectrum singularity ensemble are*

$$\theta_{0\mathrm{V}} = \mu + \bar{\omega}, \qquad \theta_{1\mathrm{V}} = -\mu - \omega, \qquad \theta_{\infty\mathrm{V}} = 2\mu - 2\omega_1, \qquad \theta_6 = -2\omega_1. \tag{6.18}$$

The Stokes multipliers are

$$\hat{s}_0 = \frac{2\pi\mathrm{i}}{\Gamma(1-2\omega_1)\Gamma(1+2\mu)}, \qquad \hat{s}_1 = -\frac{2\pi\mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}(2\mu-2\omega_1)}}{\Gamma(2\omega_1)\Gamma(-2\mu)}. \tag{6.19}$$

The monodromy matrices are

$$\hat{M}_{0\mathrm{V}} = \begin{pmatrix} \mathrm{e}^{\pi\mathrm{i}(\mu+\bar{\omega})}(1-\xi^*) - \mathrm{e}^{\pi\mathrm{i}(\mu-\bar{\omega})}2\mathrm{i}\sin\pi 2\mu & -\frac{\Gamma(1+2\mu)}{\Gamma(2\omega_1)}\mathrm{e}^{\mathrm{i}\pi(2\mu+2\omega_1)}\left[\mathrm{e}^{-\pi\mathrm{i}(\mu+\bar{\omega})}2\mathrm{i}\sin\pi 2\mu + \mathrm{e}^{\pi\mathrm{i}(-\mu+\bar{\omega})}\xi^*\right] \\ \frac{\Gamma(2\omega_1)}{\Gamma(1+2\mu)}\mathrm{e}^{\pi\mathrm{i}(2\mu-2\omega_1)}\left[2\mathrm{i}\sin\pi(\mu-\bar{\omega}) + \mathrm{e}^{\pi\mathrm{i}(-\mu+\bar{\omega})}\xi^*\right] & \mathrm{e}^{\pi\mathrm{i}(3\mu-\bar{\omega})} + \mathrm{e}^{\pi\mathrm{i}(\mu+\bar{\omega})}\xi^* \end{pmatrix}, \tag{6.20}$$

$$\hat{M}_{1\mathrm{V}} = \begin{pmatrix} \mathrm{e}^{\pi\mathrm{i}(\mu+\omega)} + \mathrm{e}^{-\pi\mathrm{i}(\mu+\omega)}\xi^* & \frac{\Gamma(1+2\mu)}{\Gamma(2\omega_1)}\,\mathrm{e}^{\pi\mathrm{i}(-\mu+\bar{\omega})}\xi^* \\ -\frac{\Gamma(2\omega_1)}{\Gamma(1+2\mu)}\,\mathrm{e}^{-\pi\mathrm{i}2\omega_1}[2\mathrm{i}\sin\pi(\mu+\omega) + \mathrm{e}^{-\pi\mathrm{i}(\mu+\omega)}\xi^*] & \mathrm{e}^{-\pi\mathrm{i}(\mu+\omega)}(1-\xi^*) \end{pmatrix}, \tag{6.21}$$

$$\hat{M}_{\infty\mathrm{V}} = \begin{pmatrix} \mathrm{e}^{\pi\mathrm{i}(2\mu-2\omega_1)} & -\frac{2\pi\mathrm{i}}{\Gamma(-2\mu)\Gamma(2\omega_1)} \\ \frac{2\pi\mathrm{i}}{\Gamma(1+2\mu)\Gamma(1-2\omega_1)}\,\mathrm{e}^{\pi\mathrm{i}(2\mu-2\omega_1)} & 2\cos\pi(2\mu+2\omega_1) - \mathrm{e}^{\pi\mathrm{i}(2\mu-2\omega_1)} \end{pmatrix}. \tag{6.22}$$

Proof. The set of $\mathrm{P_{VI}}$ parameters (3.4) in our application implies that the limiting $\mathrm{P_V}$ parameters are $\theta_6 = -2\omega_1, \theta_{0\mathrm{V}} = \mu + \bar{\omega}, \theta_{1\mathrm{V}} = -\mu - \omega, \theta_{\infty\mathrm{V}} = 2\mu - 2\omega_1$, which are consistent with those of (5.8). Using the structure of the $\mathrm{P_{VI}}$ monodromy matrices (3.7) and (3.8), we compute that $\alpha = -\theta_{\infty\mathrm{V}} + \theta_6$ and $\beta = -\theta_6$, which fixes the Stokes multipliers as given by (6.19). We solve system (6.17) which yields the solution

$$K = K_{1,1}\begin{pmatrix} 1 & -\mathrm{i}\frac{m_{t\mathrm{VI}}}{2\sin\pi\theta_6} \\ \frac{2\mathrm{i}}{\hat{s}_1}\,\mathrm{e}^{\pi\mathrm{i}(\theta_{\infty\mathrm{V}}+\theta_6)}\sin\pi(-\theta_{\infty\mathrm{V}}+\theta_6) & \frac{\mathrm{e}^{\pi\mathrm{i}(\theta_{\infty\mathrm{V}}+\theta_6)}m_{t\mathrm{VI}}}{\hat{s}_1} \end{pmatrix}, \tag{6.23}$$

for an arbitrary nonzero $K_{1,1}$. Finally, we employ these findings in (6.16) and after lengthy calculations arrive at explicit forms for the monodromy matrices (6.20)–(6.22). $\square$

However, the explicit parametrization of the alternative P_V monodromy matrices (4.14), (4.15) allows us to compute these directly from the data deduced from the expansion of the τ-function about $x = 0$, as given by theorems 5.1 and 5.2.

Corollary 6.2 [2, 18]**.** *In terms of the other set of monodromy data* $\{M_{0V}, M_{1V}, M_{\infty V}\}$*, we have*

$$M_{0V} = \begin{pmatrix} e^{\pi i(-3\mu+\bar{\omega})}(1-\xi^*) & -\frac{\Gamma(1+2\mu)}{\Gamma(2\omega_1)} e^{\pi i(-\mu-\omega)}\left[2i\sin\pi 2\mu + e^{-\pi i 2\mu}\xi^*\right] \\ \frac{\Gamma(2\omega_1)}{\Gamma(1+2\mu)} e^{\pi i(-2\mu+2\omega_1)}\left[2i\sin\pi(\mu-\bar{\omega}) + e^{\pi i(-\mu+\bar{\omega})}\xi^*\right] & 2\cos\pi(\mu+\bar{\omega}) - e^{\pi i(-3\mu+\bar{\omega})}(1-\xi^*) \end{pmatrix}, \tag{6.24}$$

and

$$M_{1V} = \begin{pmatrix} e^{\pi i(\mu+\omega)} + e^{\pi i(2\omega_1-\mu+\bar{\omega})}\xi^* & \frac{\Gamma(1+2\mu)}{\Gamma(2\omega_1)} e^{\pi i(-\mu+\bar{\omega})}\xi^* \\ -\frac{\Gamma(2\omega_1)}{\Gamma(1+2\mu)} e^{\pi i 2\omega_1}[2i\sin\pi(\mu+\omega) + e^{\pi i(2\omega_1-\mu+\bar{\omega})}\xi^*] & e^{\pi i(-\mu-\omega)} - e^{\pi i(2\omega_1-\mu+\bar{\omega})}\xi^* \end{pmatrix}, \tag{6.25}$$

and

$$M_{\infty V} = \begin{pmatrix} 2\cos\pi(2\mu+2\omega_1) - e^{-\pi i(2\mu-2\omega_1)} & -\frac{2\pi i}{\Gamma(-2\mu)\Gamma(2\omega_1)} \\ \frac{2\pi i}{\Gamma(1+2\mu)\Gamma(1-2\omega_1)} e^{-\pi i(2\mu-2\omega_1)} & e^{-\pi i(2\mu-2\omega_1)} \end{pmatrix}. \tag{6.26}$$

Proof. The first two matrices are computed from (4.14) and (4.15) after noting that a comparison of the Stokes multipliers (4.17) and (6.19) obliges us to set $r = -2\mu$. The third matrix is computed using (4.7). As a check, one can verify directly that both sets of monodromy matrices (6.20)–(6.22) and (6.24)–(6.26) are related by the transformations of (4.9). □

In the discussion of the asymptotics of the P_V σ-function as $t \to \pm\infty$ given in [2] the following parameter

$$\beta_0 = \frac{1}{2\pi i}\ln\{\xi^*[1 - e^{\pi i(-2\mu+2\omega_1)}(1-\xi^*)]\} \tag{6.27}$$

was found to be crucial. We note that when $\xi^* = 1$, both β_0 and the upper left element of M_0 vanish. A comparison of our monodromy matrices, evaluated at $\xi^* = 1$, with those discussed in [1] where a solution to one of the connection problems was reported reveals that our case is precisely the case of the *lower truncated solution*. In terms of the parameters used in that work we have

$$i\hat{u} = 2^{2(2\mu-2\omega_1)} e^{-\pi i(\mu-\bar{\omega})} \frac{\Gamma(1+2\mu)}{\Gamma(2\omega_1)}, \tag{6.28}$$

$$i\hat{v} = \sqrt{\frac{2}{\pi}} e^{\pi i(\mu+\omega_1)} \cos\pi(\mu-\bar{\omega}). \tag{6.29}$$

This means that the σ-function has the following asymptotic expansion:

$$\zeta(s) \underset{s\to -i\infty}{\sim} \zeta_0(s) - i\, e^{\pi i(\mu+\omega_1)}\cos\pi(\mu-\bar{\omega})\sqrt{\frac{|s|}{4\pi}}\, e^{-is/2} \tag{6.30}$$

in the sector $-\pi \leqslant \arg(s) \leqslant 0$. Here $\zeta_0(s)$ is the formal, algebraic expansion

$$\begin{aligned} \zeta_0(s) \underset{|s|\to\infty}{\sim} & \frac{s^2}{16} + \left(\mu - \frac{1}{2}i\omega_2\right)s + 4\mu^2 - 2\mu i\omega_2 + \omega_1^2 + \omega_2^2 - \frac{1}{4} \\ & - 2i\omega_2\left(4\mu^2 - 4\omega_1^2\right)s^{-1} + \left[16\mu^4 - 8\left(4\left(\omega_1^2+\omega_2^2\right)+1\right)\mu^2 - 16\omega_1^2\omega_2^2 \right. \\ & \left. + \left(4\omega_1^2 - 1\right)\left(4\omega_1^2 - 4\omega_2^2 - 1\right)\right]s^{-2} + O(s^{-3}). \end{aligned} \tag{6.31}$$

However, this is not the physically interesting asymptotic expansion of $s \to +i\infty$.

7. Conclusions

An especially interesting problem that remains outstanding is the connection formula relating the $x \to 0$ behaviour of the $\mathrm{P_V}$ τ-function arising in the bulk scaling of the spectrum singularity ensemble $\mathcal{A}(x)$, as given in (5.7), to the $x \to +\mathrm{i}\infty$ behaviour. The bulk scaling limit of the partition function for the Dyson CUE (3.1) is the generating function for the gap probability, which has the Fredholm determinant formula

$$E((-t,t);\xi) = \det(\mathbb{I} - \xi\mathbb{K}|_{L^2(-t,t)}), \tag{7.1}$$

where the integral operator $\mathbb{K}$ has a kernel with the sine kernel form

$$K(x,y) = \frac{\sin\pi(x-y)}{\pi(x-y)}. \tag{7.2}$$

This is a special case of our spectral average $\mathcal{A}(x)$ and the precise relation is

$$E((-t,t);\xi) = \mathcal{A}(4\mathrm{i}t;\mu=0,\omega_1=0,\omega_2=0;\xi). \tag{7.3}$$

The large $t \to \infty$ asymptotic expansion of the gap probability $E((-t,t);\xi=1)$ is known to be

$$E((-t,t);\xi=1) \sim \mathrm{e}^{3\zeta'(-1)+\frac{1}{12}\log 2} t^{-1/4}\,\mathrm{e}^{-\frac{1}{2}t^2+\mathrm{o}(1)}, \tag{7.4}$$

or alternatively as

$$t\frac{\mathrm{d}}{\mathrm{d}t}\log E((-t,t);\xi=1) \sim -t^2 - \frac{1}{4} - \frac{1}{16t^2} - \frac{5}{32t^4} + \cdots. \tag{7.5}$$

For $0 < \xi < 1$, this result becomes

$$t\frac{\mathrm{d}}{\mathrm{d}t}\log E((-t,t);\xi) \sim \frac{2t}{\pi}\log(1-\xi) + \frac{\log^2(1-\xi)}{2\pi^2} + \mathrm{o}(1). \tag{7.6}$$

This asymptotic problem has been the subject of intense and continuing study [3, 4, 7–11, 21, 22]. The question we pose then is: what is the generalization of this asymptotic result valid for generic ω_1, ω_2, μ and that is uniformly valid for $|1-\xi| < \delta$ with some finite $\delta > 0$?

Our results do not strictly apply for the situation of moments or singularities that are negative $\Re(\mu), \Re(\omega_1) < -1/2$ and $|t| \to 1$, however the present results may carry over to this case. This case is quite relevant in the context of studies concerning the averages of the characteristic polynomial in the CUE with negative integer powers [12] where such averages are employed to make conjectures concerning the averages of powers of the zeta or L-functions on their critical lines.

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